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Some Inequalities for the Forward Scattering Amplitude*

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We develop a class of inequalities involving the real parts of the forward scattering amplitude at arbitrary energies and certain finite integrals over total cross sections. These should prove useful in the resolution of phase shift ambiguities in the analyses of scattering data and are sufficiently flexible to be applied even in situations where, for instance, the magnitude of the residue at a pole is not known.

I. INTRODUCTION

SINCE the first development of the subject by Goldberger and others,[1] forward dispersion relations have been extensively used in the phenomenological analyses of π-N,[2] K-N,[3] and N-N[4] scattering data. The technique has had a somewhat limited success, however, due to the presence of three adverse factors: (a) ignorance of total cross sections in the high-energy region, (b) ignorance of the magnitudes of the residues at certain poles, and (c) the presence of unphysical regions in the dispersion integrals. The dispersion relations have hitherto been used as identities for the real parts of the forward scattering amplitudes in terms of certain integrals over total cross sections. We wish to demonstrate in this paper that if these are converted into suitable inequalities, the factors (a) and (b) need no longer be problems while the difficulties associated with the factor (c) can be rendered much less severe. It is, however, true that in this process, we lose some of the information which is in principle contained in the canonical identities.

Section II begins with a derivation of these inequalities for $\pi^{\pm}-p$ scattering under very weak assumptions regarding the asymptotic behavior of the amplitude. These results are then extended by observing that the $\pi^{-}-p$ total cross section seems to be always somewhat larger than the $\pi^{+}-p$ total cross section beyond a certain energy. Still further inequalities can be proved if we are willing to assume that the real part of the amplitude does not increase like energy itself at large energies. Evidence for this assumption is somewhat ambiguous, but it seems intuitively rather plausible.

Section III is concerned with the $K^{\pm}-p$ system which is a typical example where both unknown coupling constants and unphysical regions occur. We show how inequalities can be written down which do not involve coupling constants provided their signs are known. These signs depend essentially only on the relative parities of the particles involved and are rather well established for this system.[5] The section concludes with some remarks on the unphysical region.

The Appendix describes how one can derive a whole class of simpler inequalities from the previous results and follows the discussion of a similar problem by the author in a different context.[6]

The extension of the foregoing considerations to other scattering processes is fairly trivial and is not therefore considered in this paper.

For the benefit of the reader interested primarily in the results, we mention that these are contained in Eqs. (II.9), (II.16), (II.23), (III.4), (III.6)–(III.10), and the Appendix. The principal conventions about

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[1] M. Gell-Mann, M. L. Goldberger, and W. Thirring, Phys. Rev. **95**, 1612 (1954); M. L. Goldberger, *ibid.* **99**, 979 (1955); M. L. Goldberger, H. Miyazawa, and R. Oehme, *ibid.* **99**, 986 (1955).

[2] H. L. Anderson, W. C. Davidon, and U. E. Kruse, Phys. Rev. **100**, 339 (1955); U. Haber-Schaim, *ibid.* **104**, 1113 (1956); T. Spearman, Nucl. Phys. **16**, 402 (1960); F. Salzman and G. Salzman, Phys. Rev. **120**, 599 (1960). See G. Puppi, *Proceedings of the 1958 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1958), p. 43 for further references.

[3] See R. H. Dalitz, *Proceedings of the 1958 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1958), p. 191 and R. Karplus, L. Kerth and T. Kycia, Phys. Rev. Letters **2**, 510 (1959).

[4] Riazuddin, Phys. Rev. **121**, 1509 (1961).

[5] A summary of the evidence and detailed references are to be found in G. A. Snow, *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 795.

[6] A. P. Balachandran, EFINS-63-67 (unpublished); J. Math. Phys. (to be published).

notation are explained at the beginning of Secs. II and III, in the three paragraphs which follow Eq. (II.8) and in the ones which include Eqs. (II.11) and (III.7).

II. THE π-p FORWARD SCATTERING AMPLITUDE

Let ω and k denote the laboratory energy and momentum of the pion and let m_π and m_N denote the pion and nucleon masses. The twice subtracted forward dispersion relation is then

$$\mathrm{Re}f_-(\omega)=\frac{2f^2}{\omega+\omega_N}+a+b\omega+\frac{\omega^2}{\pi}P\int_{m_\pi}^{\infty}\frac{d\omega'}{\omega'^2}\times\left[\frac{\mathrm{Im}f_-(\omega')}{\omega'-\omega}+\frac{\mathrm{Im}f_+(\omega')}{\omega'+\omega}\right],\quad \omega\geq m_\pi, \tag{II.1}$$

where $f_\mp(\omega)$ are the $\pi^\mp-p$ forward scattering amplitudes, $f^2=0.08$ and $\omega_N=m_\pi^2/2m_N$. For (II.1) to be true, it is sufficient that $f_-(\omega)$ be $O(\omega^{1+\delta}\ln^\alpha\omega)$ for $\delta<1$ and for some α as $\omega\to\infty$ in any direction. If $\sigma_\mp(\omega)$ are the total cross sections, $f_\mp(\omega)$ are normalized such that

$$\mathrm{Im}f_\mp(\omega)=(k/4\pi)\sigma_\mp(\omega),\quad \omega\geq m_\pi. \tag{II.2}$$

The symmetry relation

$$f_-(-\omega)=f_+(\omega) \tag{II.3}$$

is also necessary to derive (II.1).

The symbols O and o will often be used in this paper in certain well-known senses. Thus, $f_-(\omega)=O(\omega)$ as $\omega\to a$ means that $|f_-(\omega)|<C|\omega|$ for some fixed C as $\omega\to a$ while $f_-(\omega)=o(\omega)$ as $\omega\to a$ means that $|f_-(\omega)|<\epsilon|\omega|$ for any $\epsilon>0$ as $\omega\to a$. The latter in particular can also be written as $f_-(\omega)/\omega\to 0$ as $\omega\to a$.

We will derive inequalities for the values of $f_-(\omega)$ only at energies with an immediate experimental significance. It is easy enough, however, to extend these results into inequalities involving its value at any real or complex point by techniques similar to those described in Ref. 6.

Let ω_i $(i=1,2,\cdots,p)$ be any p distinct points at which $\mathrm{Re}f_-(\omega)$ is known through, for instance, a phase shift analysis and let ω_i $(i=p+1, p+2,\cdots,n)$ be the *negatives* of any n-p points at which $\mathrm{Re}f_+(\omega)$ is known. Further, let E_i $(i=1,2,\cdots,n-3)$ be some real variables. Define

$$g_-(\omega)=f_-(\omega)\frac{\prod_{i=1}^{n-3}(\omega-E_i)}{\prod_{i=1}^{n}(\omega-\omega_i)},\quad n\geq 3 \tag{II.4}$$

where the product in the numerator is to be set equal to one if $n=3$. We assume that $f_-(\omega)=O(\omega^{2+\delta}\ln^\alpha\omega)$ for $\delta<1$ and for some α as $\omega\to\infty$ in any direction. For real directions, this is almost certainly true, being much weaker than the Froissart bound.[7] The dispersion relation for $g_-(\omega)$ with these assumptions reads

$$\mathrm{Re}g_-(\omega)=-\frac{2f^2}{(\omega+\omega_N)}\frac{\prod(\omega_N+E_i)}{\prod(\omega_N+\omega_i)}+\sum_j\frac{\mathrm{Re}f_-(\omega_j)}{(\omega-\omega_j)}\frac{\prod_i(\omega_j-E_i)}{\prod_{i\neq j}(\omega_j-\omega_i)}$$
$$+\frac{P}{\pi}\int_{m_\pi}^{\infty}d\omega'\left[\frac{\mathrm{Im}f_-(\omega')}{(\omega'-\omega)}\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\frac{\mathrm{Im}f_+(\omega')}{(\omega'+\omega)}\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}\right],\quad \omega\geq m_\pi, \tag{II.5}$$

where the principal value is to be taken at each of the ω_i and at ω. The relative minus sign of the two terms in the integral has come about from the identity

$$\frac{\prod_{i=1}^{n-3}(-\omega'-E_i)}{\prod_{i=1}^{n}(-\omega'-\omega_i)}=-\frac{\prod_{i=1}^{n-3}(\omega'+E_i)}{\prod_{i=1}^{n}(\omega'+\omega_i)} \tag{II.6}$$

and is important for our analysis.

The Froissart bound[7] also implies that $\mathrm{Re}f_-(\omega)=o(\omega^2)$ as $\omega\to\infty$ along the real axis. Therefore, $g_-(\omega)=o(1/\omega)$ or the coefficient of $1/\omega$ on the right side of (II.5) must vanish as $\omega\to\infty$. That is,

$$-2f^2\frac{\prod(\omega_N+E_i)}{\prod(\omega_N+\omega_i)}+\sum_j\mathrm{Re}f_-(\omega_j)\frac{\prod_i(\omega_j-E_i)}{\prod_{i\neq j}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\infty}d\omega'\left[\mathrm{Im}f_-(\omega')\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}+\mathrm{Im}f_+(\omega')\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}\right]=0. \tag{II.7}$$

[7] M. Froissart, Phys. Rev. **123**, 1053 (1961); A. Martin, *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 566.

At the negative ω_j, $\mathrm{Re}f_-(\omega_j)$ can be expressed in terms of $\mathrm{Re}f_+(-\omega_j)$ through the symmetry relation (II.3). The sum rule (II.7) thus involves only measurable quantities. For sufficiently large ω', each term of the integrand is positive definite since $\mathrm{Im}f_\mp(\omega')$ is so because of (II.2). Therefore, cancellation between these two terms cannot be responsible for the convergence of the integrals. But since the integrals must necessarily exist if what we have assumed about $\mathrm{Re}f_-(\omega)$ is correct, $\mathrm{Im}f_\mp(\omega)$ must in fact be $o(\omega^2/\ln\omega)$ and not merely $O(\omega^{2+\delta}\ln^\alpha\omega)$ as $\omega\to\infty$ along the real axis. We will often encounter this type of result in our subsequent analysis.[8]

Let us now define two real energies ω_r and ω_l such that $\omega_r>\max\omega_i$ $(1=1,2,\cdots,p)$ and $\omega_l>\max|\omega_i|$ $(i=p+1, p+2,\cdots,n)$. We can then rewrite (II.7) as

$$-2f^2\frac{\prod(\omega_N+E_i)}{\prod(\omega_N+\omega_i)}+\sum_i \mathrm{Re}f_-(\omega_j)\frac{\prod_i(\omega_j-E_i)}{\prod_{i\neq j}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_{m\pi}^{\omega_r}d\omega'\,\mathrm{Im}f_-(\omega')\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m\pi}^{\omega_l}d\omega'\,\mathrm{Im}f_+(\omega')\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}$$
$$=\frac{1}{\pi}\int_{\omega_r}^{\infty}d\omega'\,\mathrm{Im}f_-(\omega')\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}+\frac{1}{\pi}\int_{\omega_l}^{\infty}d\omega'\,\mathrm{Im}f_+(\omega')\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}. \quad \text{(II.8)}$$

It is convenient to introduce a few definitions at this point: (a) For any real a and b with $a\leq b$, $\{E_i\}\in D(a,b)$ if some pairs of E_i are equal and assume any real value and the rest are constrained by the inequality $a\leq E_i\leq b$. (b) $\omega_r\in E_p(\omega_q)$ if $\omega_r>\max\omega_i$ $(i=q, q+1,\cdots,q+p)$. (c) For a real w, $\omega_r\in F_p(\omega_q,w)$ if $\omega_r>\max\omega_i$ $(i=q,q+1,\cdots,q+p)$ or $\omega_r\geq w$ according as whether $\max\omega_i>w$ or $w>\max\omega_i$ for $i=q,q+1,\cdots,q+p$. In this case, if $\max\omega_i=w$, ω_r should be $>w$. The corresponding notation for ω_l can be illustrated by observing that in (II.8), $\omega_l\in E_{n-p}(|\omega_{p+1}|)$. (d) For a real μ, $\{\omega_i\}\in R(\mu)$ if all the ω_i are $\geq\mu$. (e) $\{\omega_i\}\in\bar{R}(\mu)$ if all the ω_i are $>\mu$. (f) $\{\omega_i\}\in S(-\mu)$ if all the ω_i are $\leq-\mu$.

The definitions (a), (d), (e), and (f) will also be used for the individual ω_i. Thus, for instance, if only the first p of the ω_i's are $\geq\mu$, we will write $\omega_i\in R(\mu)$ $(i=1,2,\cdots,p)$.

The contents of this notation will perhaps be clarified if it is observed that if all the ω_i are π^--p points, $\{\omega_i\}\in R(m_\pi)$while if only the first p of them are π^--p points and the rest negatives of π^+-p points, $\omega_i\in R(m_\pi)$ for $i=1,2,\cdots,p$ and $\omega_i\in S(-m_\pi)$ for $i=p+1,p+2,\cdots,n$.

The domains of ω_r and ω_l have been so defined in (II.8) as to render the products $\Pi(\omega'-\omega_i)$ and $\Pi(\omega'+\omega_i)$ positive definite on its right side. Therefore, since $\mathrm{Im}f_\mp(\omega)$ are also positive definite, the right side as a whole is so when $\{E_i\}\in D(-\omega_l,\omega_r)$. That is,

$$-2f^2\frac{\prod_{i=1}^{n-3}(\omega_N+E_i)}{\prod_{i=1}^{n}(\omega_N+\omega_i)}+\sum_{j=1}^{n}\mathrm{Re}f_-(\omega_j)\frac{\prod_{i=1}^{n-3}(\omega_j-E_i)}{\prod_{i\neq j}^{n}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_{m\pi}^{\omega_r}d\omega'\mathrm{Im}f_-(\omega')\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod_{i=1}^{n}(\omega'-\omega_i)}$$
$$-\frac{P}{\pi}\int_{m\pi}^{\omega_l}d\omega'\,\mathrm{Im}f_+(\omega')\frac{\prod_{i=1}^{n-3}(\omega'+E_i)}{\prod_{i=1}^{n}(\omega'+\omega_i)}>0 \quad \text{(II.9)}$$

if the first p of the ω_i's $\in R(m_\pi)$, the rest of the ω_i's $\in S(-m_\pi)$, $\omega_r\in E_p(\omega_1)$, $\omega_l\in E_{n-p}(|\omega_{p+1}|)$ and $\{E_i\}\in D(-\omega_l,\omega_r)$. This is the first of the promised inequalities and has the interesting feature that it allows the use of π^--p and π^+-p data simultaneously. Incidentally, it is not permitted to set both ω_r and ω_l equal to infinity in (II.9), for then, we get back (II.7). The inequality of course gets stronger with increasing ω_r and ω_l.

The reason why we arranged for a minus sign between the two terms of the integral in (II.5) should now be clear. If it had been a plus, we could not have got any inequality involving only finite integrals without extra assumptions regarding the relative magnitudes of the two total cross sections. This last observation will be developed further presently. [Cf. Eq. (II.16) with condition (a) and $E_{n-3}=-\infty$.]

[8] See, in this connection, D. Amati, M. Fierz, and V. Glaser, Phys. Rev. Letters **4**, 89 (1960); M. Sugawara and A. Kanazawa, Phys. Rev. **123**, 1895 (1961); S. Weinberg, *ibid.* **124**, 2049 (1961).

It is instructive to rewrite (II.9) in terms of $f_+(\omega)$:

$$-2f^2\frac{\prod(\omega_N-E_i)}{\prod(\omega_N-\omega_i)}+\sum_i \operatorname{Re}f_+(\omega_j)\frac{\prod_i(\omega_j-E_i)}{\prod_{i\neq j}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_r}d\omega' \operatorname{Im}f_+(\omega')$$
$$\times\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_l}d\omega' \operatorname{Im}f_-(\omega')\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}>0 \quad \text{(II.10)}$$

when $\omega_r\in E_p(\omega_1)$, $\omega_l\in E_{n-p}(|\omega_{p+1}|)$ and $\{E_i\}\in D(-\omega_l,\omega_r)$. The ω_i ($i=1, 2, \cdots, p$) now correspond to π^+-p points and the rest of the ω_i correspond to negatives of the π^--p points.

To proceed further, let us assume, as is indicated by experiments, that $\sigma_-(\omega)\geq\sigma_+(\omega)$ for $\omega\geq\omega_0$ where ω_0 is to be determined empirically. Let

$$\operatorname{Im}f_-(\omega)=\operatorname{Im}f_+(\omega)+\varphi_-(\omega)\,, \quad \text{(II.11)}$$

where $\varphi_-(\omega)\geq0$ for $\omega\geq\omega_0$. In the first instance, let $\{\omega_i\}\in R(m_\pi)$, that is, let all of them be π^--p points. We can rewrite (II.7) in the form

$$-2f^2\frac{\prod(\omega_N+E_i)}{\prod(\omega_N+\omega_i)}+\sum_i \operatorname{Re}f_-(\omega_j)\frac{\prod_i(\omega_j-E_i)}{\prod_{i\neq j}(\omega_j-\omega_i)}$$
$$-\frac{P}{\pi}\int_{m_\pi}^{\omega_r}d\omega'\varphi_-(\omega')\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_l}d\omega' \operatorname{Im}f_+(\omega')\left[\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}+\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}\right]$$
$$=\frac{1}{\pi}\int_{\omega_r}^{\infty}d\omega'\phi_-(\omega')\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}+\frac{1}{\pi}\int_{\omega_l}^{\infty}d\omega' \operatorname{Im}f_+(\omega')\left[\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}+\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}\right], \quad \text{(II.12)}$$

where $\omega_r\in F_n(\omega_1,\omega_0)$ and $\omega_l\in E_n(\omega_1)$. The first integral on the right side of (II.12) is positive if $\{E_i\}\in D(-\infty,\omega_r)$. Since all the ω_i are positive and $\omega_l>\max\omega_i$, it is true that

$$\frac{1}{\prod(\omega'-\omega_i)}>\frac{1}{\prod(\omega'+\omega_i)}>0. \quad \text{(II.13)}$$

Since also

$$\prod(\omega'-E_i)>0, \quad \text{(II.14)}$$
$$\prod(\omega'-E_i)+\prod(\omega'+E_i)>0,$$

when $E_i\leq0$ for every i, we have

$$\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}+\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}>\frac{1}{\prod(\omega'+\omega_i)}[\prod(\omega'-E_i)+\prod(\omega'+E_i)]>0 \quad \text{(II.15)}$$

when $E_i\leq0$. Putting all these together, we find

$$-2f^2\frac{\prod_{i=1}^{n-3}(\omega_N+E_i)}{\prod_{i=1}^{n}(\omega_N+\omega_i)}+\sum_{j=1}^{n}\operatorname{Re}f_-(\omega_j)\frac{\prod_{i=1}^{n-3}(\omega_j-E_i)}{\prod_{i\neq j}^{n}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_r}d\omega'\varphi_-(\omega')\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod_{i=1}^{n}(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_l}d\omega' \operatorname{Im}f_+(\omega')$$
$$\times\left\{\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod_{i=1}^{n}(\omega'-\omega_i)}+\frac{\prod_{i=1}^{n-3}(\omega'+E_i)}{\prod_{i=1}^{n}(\omega'+\omega_i)}\right\}>0 \quad \text{(II.16)}$$

if any one of the following conditions are satisfied: (a) All the $\omega_i\in R(m_\pi)$, $\omega_r\in F_n(\omega_1,\omega_0)$, $\omega_l\in E_n(\omega_1)$ and $E_i\leq0$. (b) The first p of the $\omega_i\in R(m_\pi)$, the rest of the $\omega_i\in S(-m_\pi)$, the choice of the ω_i is such that (II.13) holds for $\omega'\geq\omega_l$, $\omega_r\in F_p(\omega_1,\omega_0)$, $\omega_l\in E_n(|\omega_1|)$ and $E_i\leq0$. (c) $\omega_r\in F_p(\omega_1,\omega_0)$ and $\omega_l\in E_n(|\omega_1|)$, and $\{E_i\}\in D(-\omega_l,\omega_l)$ or

$\{E_i\}\in D(-\omega_l,\omega_r)$ according as whether $\omega_l\leq\omega_r$ or $\omega_l>\omega_r$. The $\{\omega_i\}$ here are the usual mixture of $\pi^{\mp}-p$ points. This last condition follows directly from (II.12) while (b) is evident from the derivation leading to (II.16).

It may be useful to observe that we do not really require $\varphi_-(\omega)$ to be positive definite for all $\omega\geq\omega_0$. It is sufficient if

$$\frac{1}{\pi}\int_{\omega_r}^{\infty}d\omega'\,\varphi_-(\omega')\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}\geq 0 \tag{II.17}$$

for the appropriate values of E_i, or in case it is negative, if the second integral in (II.12) dominates the first for these E_i. Equation (II.17) looks rather plausible since at the observed energies $\geq\omega_0$, $\varphi_-(\omega)$ is positive while the contribution from a possible change of sign of $\varphi_-(\omega)$ at higher energies tends to get suppressed by the extra factor $1/\omega'^3$ in the integrand. If desired, the high-energy contribution can be suppressed further by reducing the number of E_i to $n-3-2q$ for some integer q. This will change the factor $1/\omega'^3$ to $1/\omega'^{3+2p}$ while leaving the inequalities unaltered.

Observations seem to reveal that total cross sections approach constants at high energies which implies that $\mathrm{Im}f_{\mp}(\omega)\sim(\omega/4\pi)\sigma_{\mp}(\infty)$ for large ω. Due to the incoherence of many phase shifts and to the dominance of inelastic channels, we would also expect $\mathrm{Re}f_{\mp}(\omega)/\mathrm{Im}f_{\mp}(\omega)$ or equivalently, $\mathrm{Re}f_{\mp}(\omega)/\omega$ to tend to zero with increasing energy. Slightly weaker assumptions than these are sufficient for our purposes. Thus we shall assume that $f_-(\omega)=O(\omega^{1+\delta}\ln^{\alpha}\omega)$ for $\delta<1$ and for some α as $\omega\to\infty$ in any direction and $\mathrm{Re}f_-(\omega)/\omega\to 0$ as $\omega\to\infty$ along the real axis.[9] If we choose $n-2$, instead of $n-3$, E_i's with $n\geq 2$, (II.7) is seen to be replaced by

$$2f^2\frac{\prod(\omega_N+E_i)}{\prod(\omega_N+\omega_i)}+\sum_i \mathrm{Re}f_-(\omega_j)\frac{\prod_i(\omega_j-E_i)}{\prod_{i\neq j}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\infty}d\omega'\left[\mathrm{Im}f_-(\omega')\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\mathrm{Im}f_+(\omega')\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}\right]=0, \tag{II.18}$$

where the products in the numerators are as usual to be set equal to one if $n=2$. Write this equation in the form

$$\begin{aligned}&2f^2\frac{\prod(\omega_N+E_i)}{\prod(\omega_N+\omega_i)}+\sum_i \mathrm{Re}f_-(\omega_j)\frac{\prod_i(\omega_j-E_i)}{\prod_{i\neq j}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_r}d\omega'\,\varphi_-(\omega')\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_l}d\omega'\,\mathrm{Im}f_+(\omega')\\&\quad\times\left[\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}\right]=\frac{1}{\pi}\int_{\omega_r}^{\infty}d\omega'\,\varphi_-(\omega')\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}+\frac{1}{\pi}\int_{\omega_l}^{\infty}d\omega'\,\mathrm{Im}f_+(\omega')\\&\qquad\times\left[\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}\right].\end{aligned} \tag{II.19}$$

With the stated assumptions, the two infinite integrals in (II.19) will exist.[8] Let $\{\omega_i\}\in R(m_\pi)$ and $\omega_{r,l}\in E_n(\omega_1)$. Since

$$\prod(\omega'-E_i)-\prod(\omega'+E_i)\geq 0$$

and

$$\prod(\omega'-E_i)>0 \tag{II.20}$$

for $E_i\leq 0$ for every i, the last integral is positive in this range of E_i because of (II.13). If we assume further that $\varphi_-(\omega)\geq 0$ for $\omega\geq\omega_0$, (II.19) yields

$$\begin{aligned}&2f^2\frac{\prod(\omega_N+E_i)}{\prod(\omega_N+\omega_i)}+\sum_i \mathrm{Re}f_-(\omega_j)\frac{\prod_i(\omega_j-E_i)}{\prod_{i\neq j}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_r}d\omega'\,\varphi_-(\omega')\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_l}d\omega'\,\mathrm{Im}f_+(\omega')\\&\qquad\times\left[\frac{\prod(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\frac{\prod(\omega'+E_i)}{\prod(\omega'+\omega_i)}\right]>0\end{aligned} \tag{II.21}$$

for $\omega_r\in F_n(\omega_1,\omega_0)$, $\omega_l\in E_n(\omega_1)$ and $E_i\leq 0$. For fixed E_i $(i=1,2,\cdots,n-3)$, the left side of (II.21) is linear in E_{n-2} and will therefore be positive for $-\infty\leq E_{n-2}\leq 0$ if it is positive at the points $E_{n-2}=-\infty$ and $E_{n-2}=0$. The point

[9] These are, however, stronger than what is required to prove the Pomeranchuk theorem, I. Ia. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. 34, 725 (1958) [English transl.: Soviet Phys.—JETP 7, 499 (1958)]. See also Ref. 8.

$E_{n-2}=-\infty$ gives us back (II.16) while $E_{n-2}=0$ gives

$$2f^2\omega_N\frac{\prod_{i=1}^{n-3}(\omega_N+E_i)}{\prod(\omega_N+\omega_i)}+\sum_j \mathrm{Re}f_-(\omega_j)\omega_j\frac{\prod_{i=1}^{n-3}(\omega_j-E_i)}{\prod_{i\neq j}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_r}d\omega'\varphi_-(\omega')\omega'\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_l}d\omega'\,\mathrm{Im}f_+(\omega)\omega'$$
$$\times\left[\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod(\omega'-\omega_i)}-\frac{\prod_{i=1}^{n-3}(\omega'+E_i)}{\prod(\omega'+\omega_i)}\right]>0 \quad \text{(II.22)}$$

when $E_i\leq 0$ ($l=1, 2, \cdots, n-3$). We should now set E_{n-3} also equal to zero since $E_{n-3}=-\infty$ gives back (II.16) with its $E_{n-3}=0$. Proceeding in this way, we finally get

$$2f^2\frac{\omega_N^{n-2}}{\prod_{i=1}^{n}(\omega_N+\omega_i)}+\sum_{j=1}^{n}\mathrm{Re}f_-(\omega_j)\frac{\omega_j^{n-2}}{\prod_{i\neq j}^{n}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_r}d\omega'\varphi_-(\omega')\frac{\omega'^{n-2}}{\prod_{i=1}^{n}(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_\pi}^{\omega_l}d\omega'\,\mathrm{Im}f_+(\omega')\omega'^{n-2}$$
$$\times\left[\frac{1}{\prod_{i=1}^{n}(\omega'-\omega_i)}-\frac{1}{\prod_{i=1}^{n}(\omega'+\omega_i)}\right]>0, \quad \text{(II.23)}$$

where $\{\omega_i\}\in R(m_\pi)$, $\omega_r\in F_n(\omega_1,\omega_0)$ and $\omega_l\in E_n(\omega_1)$. This equation is also true if the last n-p ω_i's denote negatives of the π^+-p points provided $\omega_r\in F_p(\omega_1,\omega_0)$, $\omega_l\in E_n(|\omega_1|)$ and the choice of the ω_i is such that (II.13) holds for $\omega'\geq\omega_l$.

If $\varphi_-(\omega)$ is not positive definite for all $\omega\geq\omega_0$, we require instead that

$$\frac{1}{\pi}\int_{\omega_r}^{\infty}d\omega'\varphi_-(\omega')\frac{\omega'^{n-2}}{\prod(\omega'-\omega_i)}\geq 0. \quad \text{(II.24)}$$

If it also turns out to be negative, it is sufficient if the right side of (II.19) is positive when all the E_i are zero.

III. THE K-p FORWARD SCATTERING AMPLITUDE

The K^--p forward dispersion relation reads

$$\mathrm{Re}f_-(\omega)=\frac{f_\Lambda^2}{\omega-\omega_\Lambda}+\frac{f_\Sigma^2}{\omega-\omega_\Sigma}+a+b\omega+\frac{\omega^2}{\pi}P\int_\mu^\infty\frac{d\omega'}{\omega'^2}\frac{\mathrm{Im}f_-(\omega')}{(\omega'-\omega)}+\frac{\omega^2}{\pi}P\int_{m_K}^\infty\frac{d\omega'}{\omega'^2}\frac{\mathrm{Im}f_+(\omega')}{(\omega'+\omega)},\quad \omega>\mu \quad \text{(III.1)}$$

with

$$\omega_\lambda=\frac{m_\lambda^2-m_N^2-m_K^2}{2m_N},\quad \lambda=\Lambda,\Sigma$$

and

$$\mu=\frac{(m_\Lambda+m_\pi)^2-m_N^2-m_K^2}{2m_N}. \quad \text{(III.2)}$$

The residues f_Λ^2 and f_Σ^2 at the Λ and Σ poles have positive signs since the K-Λ and the K-Σ relative parities are odd.[5]

If $f_-(\omega)=O(\omega^{2+\delta}\ln^\alpha\omega)$ for some α and for $\delta<1$ as $\omega\to\infty$ in any direction and if $\mathrm{Re}f_-(\omega)=o(\omega^2)$ as $\omega\to\infty$ along the real axis, we can prove a sum rule similar to (II.7):

$$f_\Lambda^2\frac{\prod_{i=1}^{n-3}(\omega_\Lambda-E_i)}{\prod_{i=1}^{n}(\omega_\Lambda-\omega_i)}+f_\Sigma^2\frac{\prod_{i=1}^{n-3}(\omega_\Sigma-E_i)}{\prod_{i=1}^{n}(\omega_\Sigma-\omega_i)}+\sum_{j=1}^{n}\mathrm{Re}f_-(\omega_j)\frac{\prod_{i=1}^{n-3}(\omega_j-E_i)}{\prod_{i\neq j}^{n}(\omega_j-\omega_i)}$$
$$-\frac{P}{\pi}\int_\mu^\infty d\omega'\,\mathrm{Im}f_-(\omega')\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod_{i=1}^{n}(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_K}^\infty d\omega'\,\mathrm{Im}f_+(\omega')\frac{\prod_{i=1}^{n-3}(\omega'+E_i)}{\prod_{i=1}^{n}(\omega'+\omega_i)}=0. \quad \text{(III.3)}$$

The ranges of the indices have been indicated in (III.3) as a reminder. The first p of the ω_i are taken to be $K^- - p$ points (not necessarily in the physical region) and the remaining ω_i are negatives of $K^+ - p$ points. We know that $\mathrm{Im} f_\mp(\omega)$ will vanish like k as $\omega \to m_K$ if we exclude the improbable situation where the total cross section becomes infinite at zero kinetic energy. This follows trivially from the optical theorem. However, since there is no such restriction at the point $\omega=\mu$, $\mathrm{Im} f_-(\omega)$ may not even vanish (let alone vanish sufficiently fast) as $\omega \to \mu$. The principal value integral may therefore diverge if $\omega_i=\mu$ so that we should require $\omega_i \in \bar{R}(\mu)$ for $i=1, 2, \cdots, p$. Choosing an $\omega_r \in F_p(\omega_1, m_K)$ and an $\omega_l \in E_{n-p}(|\omega_{p+1}|)$, we learn from (III.3) that

$$f_\Lambda{}^2 \frac{\prod_{i=1}^{n-3} (\omega_\Lambda - E_i)}{\prod_{i=1}^{n} (\omega_\Lambda - \omega_i)} + f_\Sigma{}^2 \frac{\prod_{i=1}^{n-3} (\omega_\Sigma - E_i)}{\prod_{i=1}^{n} (\omega_\Sigma - \omega_i)} + \sum_{j=1}^{n} \mathrm{Re} f_-(\omega_j) \frac{\prod_{i=1}^{n-3} (\omega_j - E_i)}{\prod_{i\neq j}^{n} (\omega_j - \omega_i)}$$
$$- \frac{P}{\pi} \int_\mu^{\omega_r} d\omega' \, \mathrm{Im} f_-(\omega') \frac{\prod_{i=1}^{n-3} (\omega' - E_i)}{\prod_{i=1}^{n} (\omega' - \omega_i)} - \frac{P}{\pi} \int_{m_K}^{\omega_l} d\omega' \, \mathrm{Im} f_+(\omega') \frac{\prod_{i=1}^{n-3} (\omega' + E_i)}{\prod_{i=1}^{n} (\omega' + \omega_i)} > 0 \quad \text{(III.4)}$$

when $\{E_i\} \in D(-\omega_l, \omega_r)$. The unphysical region has not been removed from (III.4), this point will be discussed later on. When the argument of $f_-(\omega)$ becomes negative, the analog of the symmetry (II.3) can be used to calculate its value.

Since the magnitudes of $f_\Lambda{}^2$ and $f_\Sigma{}^2$ are but poorly known, we shall now proceed to eliminate them from (III.4). Rewriting it in the form

$$\sum_j \mathrm{Re} f_-(\omega_j) \frac{\prod_i (\omega_j - E_i)}{\prod_{i\neq j} (\omega_j - \omega_i)} - \frac{P}{\pi} \int_\mu^{\omega_r} d\omega' \, \mathrm{Im} f_-(\omega') \frac{\prod (\omega' - E_i)}{\prod (\omega' - \omega_i)}$$
$$- \frac{P}{\pi} \int_{m_K}^{\omega_l} d\omega' \, \mathrm{Im} f_+(\omega') \frac{\prod (\omega' + E_i)}{\prod (\omega' + \omega_i)} > -f_\Lambda{}^2 \frac{\prod (\omega_\Lambda - E_i)}{\prod (\omega_\Lambda - \omega_i)} - f_\Sigma{}^2 \frac{\prod (\omega_\Sigma - E_i)}{\prod (\omega_\Sigma - \omega_i)}, \quad \text{(III.5)}$$
$$\{E_i\} \in D(-\omega_l, \omega_r),$$

we observe that we have essentially to search for ranges of E_i which will make the right side positive. Two cases are possible:

(a) $\prod (\omega_\lambda - \omega_i) > 0$, $\lambda = \Lambda, \Sigma$. In this case, the right side is positive if (i) an odd number of E_i are $\geq \omega_\Sigma$, (ii) an even number of the remaining E_i satisfy $\omega_\Lambda \leq E_i \leq \omega_\Sigma$ and (iii) the rest of the $E_i \in D(-\infty, \omega_\Lambda)$.

(b) $\prod (\omega_\lambda - \omega_i) < 0$, $\lambda = \Lambda, \Sigma$. In this case, the right side is positive if (i) an even number of E_i are $\geq \omega_\Sigma$, (ii) an even number of the remaining E_i satisfy $\omega_\Lambda \leq E_i \leq \omega_\Sigma$ and (iii) the rest of the $E_i \in D(-\infty, \omega_\Lambda)$.

Combining these with the restriction $\{E_i\} \in D(-\omega_l, \omega_r)$, we finally arrive at the inequality

$$\sum_{j=1}^{n} \mathrm{Re} f_-(\omega_j) \frac{\prod_{i=1}^{n-3} (\omega_j - E_i)}{\prod_{i\neq j}^{n} (\omega_j - \omega_i)} - \frac{P}{\pi} \int_\mu^{\omega_r} d\omega' \, \mathrm{Im} f_-(\omega') \frac{\prod_{i=1}^{n-3} (\omega' - E_i)}{\prod_{i=1}^{n} (\omega' - \omega_i)} - \frac{P}{\pi} \int_{m_K}^{\omega_l} d\omega' \, \mathrm{Im} f_+(\omega') \frac{\prod_{i=1}^{n-3} (\omega' + E_i)}{\prod_{i=1}^{n} (\omega' + \omega_i)} > 0, \quad \text{(III.6)}$$

when $\omega_r \in F_p(\omega_1, m_K)$, $\omega_l \in E_{n-p}(|\omega_{p+1}|)$ and (i) an odd (even) number of E_i are contained in $\omega_\Sigma \leq E_i \leq \omega_r$, (ii) an even number of the remaining E_i are contained in $\omega_\Lambda \leq E_i \leq \omega_\Sigma$ and (iii) the rest of the $E_i \in D(-\omega_l, \omega_\Lambda)$ if $\prod (\omega_\lambda - \omega_i) > 0 (<0)$. Here only the first p of the ω_i need belong to $\bar{R}(\mu)$. Notice that $\prod (\omega_\lambda - \omega_i) > 0$ or <0 according as whether p is even or odd.

Experimental evidence indicates that $\sigma_-(\omega) \geq \sigma_+(\omega)$ for every finite ω. This means that $\mathrm{Im} f_-(\omega) - \mathrm{Im} f_+(\omega)$

$=\varphi_-(\omega)\geq 0$ for $\omega\geq m_K$. The analog of (II.16) is therefore

$$f_\Lambda^2\frac{\prod_{i=1}^{n-3}(\omega_\Lambda-E_i)}{\prod_{i=1}^{n}(\omega_\Lambda-\omega_i)}+f_\Sigma^2\frac{\prod_{i=1}^{n-3}(\omega_\Sigma-E_i)}{\prod_{i=1}^{n}(\omega_\Sigma-\omega_i)}+\sum_{j=1}^{n}\operatorname{Re}f_-(\omega_j)\frac{\prod_{i=1}^{n-3}(\omega_j-E_i)}{\prod_{i\neq j}^{n}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_\mu^{m_K}d\omega'\operatorname{Im}f_-(\omega')\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod_{i=1}^{n}(\omega'-\omega_i)}$$
$$-\frac{P}{\pi}\int_{m_K}^{\omega_r}d\omega'\varphi_-(\omega')\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod_{i=1}^{n}(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_K}^{\omega_l}d\omega'\operatorname{Im}f_+(\omega')\left[\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod_{i=1}^{n}(\omega'-\omega_i)}+\frac{\prod_{i=1}^{n-3}(\omega'+E_i)}{\prod_{i=1}^{n}(\omega'+\omega_i)}\right]>0 \quad \text{(III.7)}$$

if any one of the following conditions are satisfied: (a) $\{\omega_i\}\in\bar{R}(\mu)$, $\omega_r\in F_n(\omega_1,m_K)$, $\omega_l\in E_n(\omega_1)$ and $E_i\leq 0$. (b) The first p of the ω_i's $\in\bar{R}(\mu)$, the rest $\in S(-m_K)$, the choice of the ω_i is such that (II.13) holds for $\omega'\geq\omega_l$, $\omega_r\in F_p(\omega_1,m_K)$, $\omega_l\in E_n(|\omega_1|)$ and $E_i\leq 0$. (c) $\omega_r\in F_p(\omega_1,m_K)$ and $\omega_l\in E_n(|\omega_1|)$, and $\{E_i\}\in D(-\omega_l,\omega_l)$ or $\{E_i\}\in D(-\omega_l,\omega_r)$ according as whether $\omega_l\leq\omega_r$ or $\omega_l>\omega_r$. Here too, only the first p ω_i need belong to $\bar{R}(\mu)$.

We can eliminate the f_Λ^2 terms easily. Thus

$$\sum_{j=1}^{n}\operatorname{Re}f_-(\omega_j)\frac{\prod_{i=1}^{n-3}(\omega_j-E_i)}{\prod_{i\neq j}^{n}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_\mu^{m_K}d\omega'\operatorname{Im}f_-(\omega')\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod_{i=1}^{n}(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_K}^{\omega_r}d\omega'\varphi_-(\omega')\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod_{i=1}^{n}(\omega'-\omega_i)}$$
$$-\frac{P}{\pi}\int_{m_K}^{\omega_l}d\omega'\operatorname{Im}f_+(\omega')\left[\frac{\prod_{i=1}^{n-3}(\omega'-E_i)}{\prod_{i=1}^{n}(\omega'-\omega_i)}+\frac{\prod_{i=1}^{n-3}(\omega'+E_i)}{\prod_{i=1}^{n}(\omega'+\omega_i)}\right]>0 \quad \text{(III.8)}$$

if (a) n is odd, $\{\omega_i\}\in\bar{R}(\mu)$, $\omega_r\in F_n(\omega_1,m_K)$, $\omega_l\in E_n(\omega_1)$ and $E_i\leq 0$, or (b) the first p of the $\omega_i\in\bar{R}(\mu)$, where p is odd, $\omega_r\in F_p(\omega_1,m_K)$, $\omega_l\in E_n(|\omega_1|)$, (II.13) is true for $\omega'\geq\omega_l$ and $E_i\leq 0$, or (c) $\omega_r\in F_p(\omega_1,m_K)$, $\omega_l\in E_n(|\omega_1|)$ and (i) an odd (even) number of E_i are contained in $\omega_\Sigma\leq E_i\leq\min\{\omega_r,\omega_l\}$, (ii) an even number of the remaining E_i are contained in $\omega_\Lambda\leq E_i\leq\omega_\Sigma$ and (iii) the rest of the $E_i\in D(-\omega_l,\omega_\Lambda)$ if $\prod(\omega_\Lambda-\omega_i)>0(<0)$.

Finally, if $f_-(\omega)=O(\omega^{1+\delta}\ln^\alpha\omega)$ for $\delta<1$ and for some α as $\omega\to\infty$ in any direction, $\operatorname{Re}f_-(\omega)=o(\omega)$ as $\omega\to\infty$ along the real axis and $\varphi_-(\omega)\geq 0$ for $\omega\geq m_K$,

$$f_\Lambda^2\frac{\omega_\Lambda^{n-2}}{\prod_{i=1}^{n}(\omega_\Lambda-\omega_i)}+f_\Sigma^2\frac{\omega_\Sigma^{n-2}}{\prod_{i=1}^{n}(\omega_\Sigma-\omega_i)}+\sum_{j=1}^{n}\operatorname{Re}f_-(\omega_j)\frac{\omega_j^{n-2}}{\prod_{i\neq j}^{n}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_\mu^{m_K}d\omega'\operatorname{Im}f_-(\omega')\frac{\omega'^{n-2}}{\prod_{i=1}^{n}(\omega'-\omega_i)}$$
$$-\frac{P}{\pi}\int_{m_K}^{\omega_r}d\omega'\varphi_-(\omega')\frac{\omega'^{n-2}}{\prod_{i=1}^{n}(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_K}^{\omega_l}d\omega'\operatorname{Im}f_+(\omega')\omega'^{n-2}\left[\frac{1}{\prod_{i=1}^{n}(\omega'-\omega_i)}-\frac{1}{\prod_{i=1}^{n}(\omega'+\omega_i)}\right]>0, \quad \text{(III.9)}$$

if $\{\omega_i\}\in\bar{R}(\mu)$, $\omega_r\in F_n(\omega_1,m_K)$ and $\omega_l\in E_n(\omega_1)$. If only the first p of the $\omega_i\in\bar{R}(\mu)$, we require instead that $\omega_r\in F_p(\omega_1,m_K)$, $\omega_l\in E_n(|\omega_1|)$ and (II.13) be true for $\omega'\geq\omega_l$.

The elimination of the f_Λ^2 terms is now trivial and the result in either case is

$$\sum_{j=1}^{n}\operatorname{Re}f_-(\omega_j)\frac{\omega_j^{n-2}}{\prod_{i\neq j}^{n}(\omega_j-\omega_i)}-\frac{P}{\pi}\int_\mu^{m_K}d\omega'\operatorname{Im}f_-(\omega')\frac{\omega'^{n-2}}{\prod_{i=1}^{n}(\omega'-\omega_i)}-\frac{P}{\pi}\int_{m_K}^{\omega_r}d\omega'\varphi_-(\omega')\frac{\omega'^{n-2}}{\prod_{i=1}^{n}(\omega'-\omega_i)}$$
$$-\frac{P}{\pi}\int_{m_K}^{\omega_l}d\omega'\operatorname{Im}f_+(\omega')\omega'^{n-2}\left[\frac{1}{\prod_{i=1}^{n}(\omega'-\omega_i)}-\frac{1}{\prod_{i=1}^{n}(\omega'+\omega_i)}\right]>0, \quad \text{(III.10)}$$

if an odd number of $\omega_i \in \bar{R}(\mu)$, the rest of the $\omega_i \in S(-m_K)$ and the restrictions on $\omega_{r,l}$ are unaltered.

As we have already observed in the previous section, the demand that $\varphi_-(\omega)$ be non-negative for $\omega \geq m_K$ can be relaxed a great deal in the inequalities starting from (III.7).

All the equations of this section involve contributions from the unphysical region $\mu \leq \omega < m_K$ of the $K^- - p$ scattering. It is unfortunately true that unitarity in the unphysical region does not impose any positive definiteness condition on $\mathrm{Im} f_-(\omega)$ and this is precisely what prevents us from getting rid of these terms. It is, however, sufficient for our purposes if we can get an idea of the signs of these integrals for an appropriate range of the E_i's and this can often be done, especially in $K^- - p$ scattering, by simple calculations involving, for instance, the known resonance contributions. Once this is done, it is usually possible to remove these terms by an appropriate choice of the parameters of the inequalities just as we removed the terms involving $f_\Lambda{}^2$ and $f_\Sigma{}^2$. We emphasize that even some error in the estimates of these signs can be tolerated. This is because our inequalities have hitherto been consequences of equalities of the form (II.8) where the right side is positive and nonzero. Therefore, when we bring over additional terms of ambiguous signs from left to right, we only require that these ambiguities do not overwhelm the quantity which was initially on the right. It is important to note that we commit ourselves much less in inequalities than in equalities and this greatly reduces the chances of mistakes. We also need know much less to use them. We pay for these advantages by losing some of the information contained in the equalities.

We remark in conclusion that it is possible to study the questions treated in this paper and in Ref. 6 in terms of certain reduced moment problems.[10] This observation will be developed further in a separate publication.

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APPENDIX

We describe how the inequalities of the previous sections can be simplified and collect all the relevant formulas in this Appendix. Much of the discussion is taken from the Appendix of Ref. 6 although the notation has been slightly altered to a form more suitable for our purposes.

After a few trivial changes of variables, any inequality of this paper can be written as

$$h_m{}^j = \frac{P}{\pi} \int_V d\omega' \psi(\omega') \omega'^j \frac{\prod_{i=1}^{m-j} (\omega' - E_i)}{\prod_{i=1}^{n} (\omega' - \omega_j)} > 0, \qquad \text{(A1)}$$

when $\{E_i\}$ lies in some real domain. The integration is always over a real range V which may possibly consist of several distinct pieces. The spectral function $\psi(\omega)$ contains δ functions to take into account the poles of $h_m{}^j$. The subscript m of $h_m{}^j$ means that it is a function of $E_1, E_2, \cdots, E_m$ which occur as the product $\prod_{i=1}^m \times (\omega' - E_i)$ in the numerator of its integrand. The superscript j means that in this product, the last j E_i's are to be set equal to zero. This index is actually zero in most of the equations we have hitherto encountered.

The following identity is often useful:

$$h_m{}^j = \sum_{k=0}^{p} (-1)^k c(p,k) h_{m-k}{}^{j+p-k}, \quad p \geq 0, \qquad \text{(A2)}$$

where $c(p,k)$ denotes the sum over all combinations of the p variables $E_{m-j-p+1}, E_{m-j-p+2}, \cdots, E_{m-j}$ taken k at a time and $c(p,0)$ is defined to be equal to one. For example,

$$\begin{aligned} h_3{}^0 &= h_3{}^1 - E_3 h_2{}^0 \quad \text{for} \quad p=1, \\ &= h_3{}^2 - (E_2+E_3) h_2{}^1 + E_2 E_3 h_1{}^0 \quad \text{for} \quad p=2, \\ &= h_3{}^3 - (E_1+E_2+E_3) h_2{}^2 + (E_1E_2+E_2E_3+E_3E_1) h_1{}^1 \\ &\qquad - E_1E_2E_3 h_0{}^0 \quad \text{for} \quad p=3. \end{aligned} \qquad \text{(A3)}$$

It may have been observed that p stands for the number of E_i's with respect to which we have expanded the integrand and then integrated term by term.

We write down two more useful identities. Thus,

$$h_m{}^j = \sum_{k=0}^{2p} \alpha_{2p-k}{}^{2p} E^{2p-k} h_{m+k-2p}{}^{j+k}, \qquad \text{(A4)}$$

where

$$\alpha_{2p-k}{}^{2p} = (-1)^k \frac{(2p)!}{k!(2p-k)!} \qquad \text{(A5)}$$

if $E_{m-j-2p+1} = E_{m-j-2p+2} = \cdots = E_{m-j} = E$. Further,

$$h_m{}^j = \sum_{k=0}^{p} \beta_{p-k}{}^p E^{2p-2k} h_{m-2p+2k}{}^{j+2k}, \qquad \text{(A6)}$$

where

$$\beta_{p-k}{}^p = (-1)^{p-k} \frac{p!}{k!(p-k)!} \qquad \text{(A7)}$$

if $E_{m-j-2p+1} = -E_{m-j-2p+2} = +E_{m-j-2p+3} = \cdots = -E_{m-j} = E$. Equations (A4) and (A6) can be proved by expanding the integrand in a binomial series.

Instead of attempting to reduce every inequality we

[10] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, 1943).

have encountered into ones with no E_i's at all, we shall discuss two representative cases $\{E_i\}\in D(a,b)$ for some real a and b and $E_i\leq 0$ for every i. It is straightforward to handle the other equations along similar lines. Let us first study the case where $\{E_i\}\in D(a,b)$. With $j=0$, (A1) then reads

$$h_m{}^0>0\,, \tag{A8}$$

when $\{E_i\}\in D(a,b)$. If we expand $h_m{}^0$ in terms of the last two E_i's, we have

$$h_m{}^0=h_m{}^2-(E_{m-1}+E_m)h_{m-1}{}^1+E_{m-1}E_mh_{m-2}{}^0\,. \tag{A9}$$

With the other variables fixed, we find that two cases are now possible: (a) $E_{m-1}=E_m$ and assumes any real value and (b) $a\leq E_{m-1}\leq b$, $a\leq E_m\leq b$. In case (a), (A8) gives

$$h_m{}^2-2E_mh_{m-1}{}^1+E_m{}^2h_{m-2}{}^0>0 \tag{A10}$$

for any real E_m. Therefore, since (A10) is quadratic in E_m,

$$\begin{aligned}&[h_{m-1}{}^1]^2<h_m{}^2\cdot h_{m-2}{}^0\,,\\ &h_m{}^2>0\,,\\ &h_{m-2}{}^0>0\,.\end{aligned} \tag{A11}$$

To reduce case (b), observe that for fixed E_m, (A9) is linear in E_{m-1} and will, therefore, be positive for $a\leq E_{m-1}\leq b$ if it is positive at the end points $E_{m-1}=a$ and $E_{m-1}=b$. That is, we require

$$\begin{aligned}&h_m{}^2-(a+E_m)h_{m-1}{}^1+aE_mh_{m-2}{}^0>0\,,\\ &h_m{}^2-(b+E_m)h_{m-1}{}^1+bE_mh_{m-2}{}^0>0\,.\end{aligned} \tag{A12}$$

But these are in turn linear in E_m and should therefore be positive for $E_m=a$ and $E_m=b$. Only the combination $E_m\neq E_{m-1}$ gives a result different from (A10) and this reads

$$h_m{}^2-(a+b)h_{m-1}{}^1+abh_{m-2}{}^0>0\,. \tag{A13}$$

In both (A11) and (A13), $E_i\in D(a,b)$ for $i=1,2,\cdots,m-3$. Note the simplification when a or b gets unbounded. Thus if $a=-\infty$, (A13) becomes

$$h_{m-1}{}^1-bh_{m-2}{}^0>0\,. \tag{A14}$$

We have not succeeded in simplifying (A11) and (A13) still further into a set of inequalities which completely exhaust them and also contain no E_i's at all. It is, however, easy to write down the inequalities implied by these equations when $a\leq E_i\leq b$ for every i. To find these equations, we have to set each of the E_i equal to a and b in turn in (A11) and (A13). This will generate a class of functions with no E_i's. We denote any member of this class by $k_m{}^i$. These can be calculated by using (A2) or by repeated application of the identity (A4). It may then be shown that (A11) and (A13) imply the following equations:

$$\begin{aligned}&[k_{m-1-2p}{}^1]^2<k_{m-2p}{}^2k_{m-2-2p}{}^0\,,\\ &k_{m-2p}{}^2>0\,,\\ &k_{m-2-2p}{}^0>0\,,\\ &k_{m-2p}{}^2-(a+b)k_{m-1-2p}{}^1+abk_{m-2-2p}{}^0>0\,,\end{aligned} \tag{A15}$$

where $p=0,1,\cdots$, integer $\{\frac{1}{2}(m-2),\frac{1}{2}(m-3)\}$. The corresponding members of the class should of course be inserted in the first and last lines of (A15). If the maximum value of p is $\frac{1}{2}(m-3)$, there will be one E_i left over in (A15) which is then to be set equal to a and b in turn. If $a=b$ the last equation of (A15) is unnecessary, being already implied by the first three. This is clear from the discussion leading to (A13).

Incidentally, (A15) contains rather more information than that implied by the variation of every E_i between a and b in (A11) and (A13). Thus, when the variable p is increased by one unit, we have effectively set a pair of E_i equal in (A11) and (A13) and made them arbitrarily large.

The case where $E_i\leq 0$ for every i is easier to discuss. Since

$$h_m{}^j \underset{E_{m-j}\to-\infty}{\longrightarrow} |E_{m-j}|h_{m-1}{}^j$$

and

$$h_m{}^j=h_m{}^{j+1} \quad\text{when}\quad E_{m-j}=0\,, \tag{A16}$$

the technique leading to (A13) shows that the necessary and sufficient conditions for (A8) to be true when $E_i\leq 0$ are the inequalities

$$h_{m-p}{}^{m-p}>0\,, \tag{A17}$$

for $p=0,1,\cdots,m$.

The reader for whose purposes the equations of this Appendix are too detailed may be able to find more convenient ones by giving the E_i's suitable values in the appropriate ranges in the inequalities given in the body of the paper.